PERFORMANCE BOUNDS FOR STAR CENTROID LOCALIZATION IN DIFFRACTION-LIMITED DIGITAL IMAGES
John A. Christian\textsuperscript{1} and Jacob Kowalski\textsuperscript{1}\textsuperscript{*}; \textsuperscript{1}Rensselaer Polytechnic Institute, Troy, NY 12180; *[chrisj9@rpi.edu]

Abstract. A brief narrative abstract describing the topic, its importance, its relationship to the field of space imaging, and key results/findings. This summary should allow a reader to determine if the rest of the extended abstract is of interest. This brief abstract must not exceed 100 words.

Introduction. Recent work has shown that autonomous spacecraft navigation may be achieved by observing the change in inter-star angle due to stellar aberration. This technique, known as StarNAV,\textsuperscript{1} requires star directions be measured to the milliarcsecond level. Such precision may be achieved using either telescopes or stellar interferometers. If the optical instrument is a telescope, we will likely be dealing with diffraction limited system. This scenario motivates the need for careful study of star centroiding performance for such systems.

In the ideal case, where there are no observable aberrations, it is possible to analytically compute the Cramer-Rao Lower Bound (CRLB) for the error in the star centroid location. This provides a firm limitation on the performance of a perfect sensor. After presenting the derivation, we assess how close to this bound one may come using a digital sensor—with a focus on spatial quantization and detector noise.

Derivation of Cramer-Rao Lower Bound of Diffraction-Limited Centroid Accuracy. Assuming a telescope with circular aperture and no optical aberrations, the diffraction pattern observed by the detector on the focal plane is the Airy pattern.\textsuperscript{2} This pattern may be interpreted as the probability density function (PDF) for where a photon will strike the focal plane (see Fig 2). The Airy pattern is given by

\[ p_\phi(\phi) = \frac{1}{\pi} \left( J_1 \left( \frac{kD\phi}{2} \right) / \phi \right)^2 \]

where \( J_1 \) is the Bessel function of the first kind of first order, \( D \) is the aperture diameter, \( \phi \) is the angle from the true star center, and \( k = 2\pi/\lambda \) is the wave number. As a PDF, the coefficient is chosen such that

\[ \int_0^{2\pi} \int_0^\infty p_\phi(\phi) \phi d\phi d\theta = 1 \]

where \( \theta \in [0, 2\pi] \). This is straightforward to check.

It is important to note that the centroiding accuracy of an isolated star is generally orders of magnitude better than the diffraction-limited resolution. The diffraction-limited resolution is generally taken to be the first dark band in the Airy pattern (the so-called Rayleigh criterion\textsuperscript{3}), which defines the minimum separation required to distinguish two sources (e.g., stars) from one another. They Rayleigh criterion is given by (see Fig. 1),

\[ \phi_{RC} = \frac{1.22 \lambda}{D} \]

and will serve as a useful reference later.

For a narrow field-of-view (FOV) telescope and a digital detector array, one finds the angle \( \phi \) is related to the distance from the true star centroid in pixels, \( \rho \), as

\[ \rho = \frac{f}{\mu} \phi \]

where \( f \) is the focal length and \( \mu \) is the pixel pitch. Substitution into Eq. 2 yields

\[ \int_0^{2\pi} \int_0^\infty p_\rho(\mu\rho/f) (\mu/f)^2 \rho d\rho d\theta = 1 \]

Consequently,

\[ p_\rho(\rho) = p_\phi(\mu\rho/f) (\mu/f)^2 = \frac{1}{\pi} \left[ J_1 (\alpha\rho) / \rho \right]^2 \]

where \( \alpha \) is defined as

\[ \alpha = \frac{\pi D\mu}{\lambda f} \]

Since the value \( \rho \) is the distance from the pattern center located at \( \{u_c, v_c\} \), the likelihood of observing a photon at any pixel coordinate \( \{u_i, v_i\} \) depends on \( \rho_i \),

\[ \rho_i = \sqrt{(u_i - u_c)^2 + (v_i - v_c)^2} \]
such that
\[
p_{\rho}(\rho_{i}|u_{c}, v_{c}) = \frac{1}{\pi} \left( \frac{J_{1}(\alpha \rho_{i})}{\rho_{i}} \right)^{2}
\]  
(9)

Now, suppose we observe \( N \) photons. The likelihood of simultaneously observing all of the \( N \) photons at their measured locations is then
\[
L = \prod_{i=1}^{N} p_{\rho}(\rho_{i}|u_{c}, v_{c})
\]  
(10)

As the joint PDF of all the photon observations, it is evident that
\[
\int \cdots \int L dA_{1} \cdots dA_{N} = 1
\]  
(11)
where \( dA_{i} \) is the differential area in pixel coordinates
\[
dA_{i} = du_{i} dv_{i} = \rho_{i} d\rho_{i} d\theta_{i}
\]  
(12)

We want to place a lower bound on the covariance for the maximum likelihood estimate (MLE) of the centroid location \( \mathbf{x}^{T} = [u_{c} \ v_{c}] \). The MLE estimate \( \hat{x} \) is the centroid location that best explains the observed pattern of photons. Thus, we find the desired covariance estimate as the Cramér-Rao bound,
\[
P_{\mathbf{x} \mathbf{x}}^{-1} \leq -E \left[ \frac{\partial^{2} \ln[L]}{\partial \mathbf{x} \partial \mathbf{x}} \right]
\]  
(13)
\[
= E \left[ \left( \frac{\partial \ln[L]}{\partial \mathbf{x}} \right)^{T} \left( \frac{\partial \ln[L]}{\partial \mathbf{x}} \right) \right]
\]  
(14)

The four terms may be considered one at a time. For example, beginning with the upper left-hand term, application of the law of the unconscious statistician (LOTUS) shows that
\[
E \left[ \left( \frac{\partial \ln[L]}{\partial u_{c}} \right)^{2} \right] = \int \cdots \int \left( \frac{\partial \ln[L]}{\partial u_{c}} \right)^{2} L dA_{1} \cdots dA_{N}
\]  
(15)

Furthermore, after compacting notation as \( p_{\rho i} = p_{\rho}(\rho_{i}|u_{c}, v_{c}) \), it is convenient to observe that
\[
\ln[L] = \sum_{i=1}^{N} \ln[p_{\rho i}]
\]  
(16)

and that
\[
\frac{\partial \ln[p_{\rho i}]}{\partial u_{c}} = \left( \frac{1}{p_{\rho i}} \right) \frac{\partial p_{\rho i}}{\partial u_{c}}
\]  
(17)

such that the expected value may be rewritten as
\[
E \left[ \left( \frac{\partial \ln[L]}{\partial u_{c}} \right)^{2} \right] = \sum_{i=1}^{N} \int \int \left[ \left( \frac{1}{p_{\rho i}} \right) \frac{\partial p_{\rho i}}{\partial u_{c}} \right]^{2} p_{\rho i} dA_{i}
\]  
(18)

Since double integral on the righthand side is the same for every measured photon (for every \( i \)), the subscript is no longer necessary as we move forward. Therefore, dropping subscript \( i \), one has
\[
E \left[ \left( \frac{\partial \ln[L]}{\partial u_{c}} \right)^{2} \right] = N \int \int \left( \frac{1}{p_{\rho}} \right) \left( \frac{\partial p_{\rho}}{\partial u_{c}} \right)^{2} p_{\rho} dA
\]  
(19)

The partial \( \partial p_{\rho}/\partial u_{c} \) may be computed as
\[
\frac{\partial p_{\rho}}{\partial u_{c}} = \frac{\partial p_{\rho}}{\partial \rho} \frac{\partial \rho}{\partial u_{c}}
\]  
(20)

such that, recalling,
\[
\frac{\partial J_{1}(\alpha x)}{\partial x} = \alpha J_{0}(\alpha x) - \frac{1}{x} J_{1}(\alpha x)
\]  
(21)
differentiation of Eq. (8) and Eq. (9) yields
\[
\frac{\partial \rho_p}{\partial u_c} = \frac{2(u - u_c)J_1(\alpha \rho)}{\pi \rho^4} [2J_1(\alpha \rho) - \alpha \rho J_0(\alpha \rho)]
\] (22)

The double integral from Eq. (19) is a little easier to solve in polar coordinates. Therefore,
\[
E \left[ \left( \frac{\partial \ln |L|}{\partial u_c} \right)^2 \right] = N \int_0^{2\pi} \int_0^{\infty} \frac{\rho}{P_{\rho \rho}} \left( \frac{\partial \rho_p}{\partial u_c} \right)^2 d\rho d\theta
\] (23)

Noting that \((u - u_c) = \rho \cos \theta\), this integral evaluates to
\[
E \left[ \left( \frac{\partial \ln |L|}{\partial u_c} \right)^2 \right] = N \alpha^2 = N \left( \frac{\pi D}{\lambda} \right)^2 \left( \frac{\mu}{f} \right)^2
\] (24)

An identical procedure produces
\[
E \left[ \left( \frac{\partial \ln |L|}{\partial v_c} \right)^2 \right] = N \alpha^2 = N \left( \frac{\pi D}{\lambda} \right)^2 \left( \frac{\mu}{f} \right)^2
\] (25)

\[
E \left[ \left( \frac{\partial \ln |L|}{\partial u_c} \right) \left( \frac{\partial \ln |L|}{\partial v_c} \right) \right] = 0
\] (26)

such that one has
\[
P_{xx}^{-1} \leq N \left( \frac{\pi D}{\lambda} \right)^2 \left( \frac{\mu}{f} \right)^2 I_{2x2}
\] (27)

which, as a diagonal matrix, produces the following covariance
\[
P_{xx} = \sigma_{uv}^2 I_{2x2}
\] (28)

where
\[
\sigma_{uv} \geq \left( \frac{\lambda}{\pi D \sqrt{N}} \right) \left( \frac{f}{\mu} \right)
\] (29)

or, with respect to bearing angle error, \(\sigma_\phi = (\mu/f)\sigma_{uv}\)
\[
\sigma_\phi \geq \frac{\lambda}{\pi D \sqrt{N}}
\] (30)

which is consistent with the relations presented in [1] and also agrees with the earlier results of Falconi and Lindgren.\(^{5,6}\)

This result may be numerically confirmed through simulation. Suppose that the location of each photon strike of the focal plane is governed by the PDF in Eq. 1. Further suppose one has a perfect sensor capable of measuring the exact focal plane coordinates of each of these photon strikes without error. In this case, the numerical variance of MLE centroid location is identical to the analytic CRLB for all values of \(N\), as is shown by the results in Fig. 3. These results are based on a 1,000-run Monte Carol analysis. Since the Airy pattern is symmetric, the centroid may also be computed as the mean value of all of the photon strike locations. The centroid computed by taking the mean is also shown in Fig. 3, which (as expected) is worse than the CRLB.

**Numerical Evaluation of Centroid Accuracy in Digital Images.** Real sensors cannot perfectly measure the coordinates of photon strikes, thus any real sensor necessarily has performance worse than the CRLB. Typical imaging systems make use of either CCD or CMOS arrays\(^1\) on the focal plane. These detectors attempt to count the photons that strike a photosensitive element (i.e., pixel) occupying a finite space on the focal plane. The size of these photosensitive elements dictates the pixel pitch and ultimately determines the amount of angular (or spatial) quantization. Oftentimes the photosensitive area of each pixel does not fill its entire image footprint (fill-factor < 1) resulting in regions within each pixel where photon strikes are not counted. Furthermore, the counting of photoelectrons generated from light striking the detector is not perfect, and this misaccounting of photon strikes also reduces centroiding performance relative to the perfect sensor (CRLB).

Therefore, the photon strikes occurring at continuous focal plane coordinates predicted by Eq. 1 are binned according to the pixel that would observe that photon. The result is a 2D array of photon strikes (e.g., as shown pictorially in Fig. 2), the detector’s shot noise— which describes both errors in counting the photoelectrons and the detector dark current—is modeled using a Poisson distribution.

In the noise-free case, we generally see performance im-
prove as the pixel pitch becomes small. To see this, define $R$ as the ratio of pixel pitch to the Rayleigh criterion (i.e., the distance from the pattern center to the first dark band of the Airy pattern in units of pixels),

$$R = \frac{f\phi_{RC}}{\mu} = 1.22\frac{\lambda f}{D\mu}$$  \hspace{1cm} (31)$$

As $R$ becomes large, both the mean and MLE solutions from the digital sensor improve, but they never quite reach their continuous sensor counterparts. This can be seen in Fig. 4.

This effect, however, is more nuanced when one considers a noisy detector. As the pixels become small compared to the Airy pattern, fewer photons strike each sensor. Consequently, for a fixed amount of detector noise, the per-pixel signal-to-noise ratio (SNR) decreases as $R$ increases. At some point the increased insight into the Airy pattern shape is overtaken by sensor noise (the per-pixel signal becomes too faint) and the error begins to increase again. Therefore, there is an optimal value of $R$ for a given SNR threshold. This can be seen in Fig. 5.

![Figure 4. Comparison of centroid errors for a perfect sensor (that measures the exact coordinates of each photon strike) and for a digital sensor with spatial quantization (pixels). These results are for $N = 10,000$.](image1)

![Figure 5. Contours of $\sigma_\phi/\phi_{RC}$ for various levels of spatial quantization and detector noise. These results are for $N = 50,000$.](image2)

References.


